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Quantum anomalies for generalized Euclidean Taub–NUT metrics

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Abstract

The generalized Taub–NUT metrics exhibit in general gravitational anomalies. This is in contrast with the fact that the original Taub–NUT metric does not exhibit gravitational anomalies, which is a consequence of the fact that it admits Killing–Yano tensors forming Stäckel–Killing tensors as products. We have found that for axial anomalies, interpreted as the index of the Dirac operator, the presence of Killing–Yano tensors is irrelevant. In order to evaluate the axial anomalies, we compute the index of the Dirac operator with the APS boundary condition on balls and on annular domains. The result is an explicit number-theoretic quantity depending on the radii of the domain. This quantity is 0 for metrics close to the original Taub–NUT metric but it does not vanish in general.

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1. Introduction

In the case of gravitational interaction, a consistent perturbative quantization is not available, even if there exist no fermions. It is of crucial importance in the construction of any quantum theory for gravitation to understand the problem of anomalies which can affect the conservation laws.

In this paper we shall investigate the quantum anomalies with regard to quadratic constants of motion in some explicit examples—the Euclidean Taub–Newman–Unti–Tamburino (Taub–NUT) space [1, 2] and its generalizations as was done by Iwai and Katayama [3–6].

Hidden symmetries are encapsulated into Stäckel–Killing (S–K) tensors, i.e. symmetric tensors $k_{\mu\nu} = k_{\nu\mu}$ satisfying the S–K equation

$$k_{(\mu\nu;\lambda)} = 0 \qquad (1)$$

where a semicolon precedes an index of covariant differentiation. For any geodesic with tangent (momentum vector) p_μ a S–K tensor generates a quadratic constant along the geodesic

$$K = k^{\mu\nu} p_\mu p_\nu, \qquad p_\mu = g_{\mu\nu}(x)\dot{x}^\nu, \tag{2}$$

where g is the metric tensor and the over-dot denotes the ordinary proper time derivative.

Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analogue of the quadratic function (2) is [7, 8]

$$\mathcal{K} = D_\mu k^{\mu\nu} D_\nu \tag{3}$$

where D_μ is the covariant differential operator on the curved manifold. Working out the commutator of (3) with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu = D_\mu g^{\mu\nu} D_\nu \tag{4}$$

and taking into account that $k_{\mu\nu}$ is a S–K tensor satisfying equation (1), we get [7]

$$[\mathcal{H}, \mathcal{K}] = -\tfrac{4}{3}\left\{k_\lambda^{[\mu} R^{\nu]\lambda}\right\}_{;\nu} D_\mu \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor. This means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [9]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S–K tensor $k_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

In general, when the manifold is not Ricci flat the operators constructed from symmetric S–K tensors are a source of gravitational anomalies for scalar fields. However, when the S–K tensors admit a decomposition in terms of antisymmetric tensors Killing–Yano (K–Y) [8] the gravitational anomaly is absent.

The K–Y tensors are profoundly connected with supersymmetric classical and quantum mechanics on curved spaces where such tensors do exist [10]. The K–Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first-order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D [8]. When the K–Y tensors enter as square roots in the structure of several second-rank S–K tensors, they generate conserved quantities in pseudo-classical models for fermions [10] or conserved operators in Dirac theory which commute with D.

In the pseudo-classical approach [10] of the fermions, the absence of the K–Y tensors hampers the evaluation of the spin contribution to the conserved quantities. Passing to the Dirac equation in a curved background, the lack of the K–Y tensors makes impossible the construction of Dirac-type operators and hidden quantum conserved operators commuting with the standard Dirac one.

Having in mind that the K–Y tensors prevent the appearance of gravitational anomalies for the scalar field and on the other hand their connection with supersymmetries and Dirac-type operators, it is natural to investigate whether they play a role also to *axial anomalies*.

The importance of anomalous Ward identities in particle physics is widely known. The anomalous divergence of the axial vector current in a background gravitational field has been largely discussed in the literature and directly related to the index theorem. In even-dimensional spaces one can define the index of a Dirac operator as the difference in the number of linearly independent zero modes with eigenvalue +1 and -1 under γ_5. The index is useful as a tool to investigate topological properties of the space, as well as in computing anomalies in quantum field theory.

Here we investigate the continuous transition from the case in which a hidden symmetry is described by a S–K tensor which can be written as a symmetrized product of K–Y tensors to the situation in which such a decomposition is not available for lack of K–Y tensors.

The standard and generalized Taub–NUT metrics are suitable for this task. Let us observe that in the standard Taub–NUT space there are no gravitational anomalies taking into account that it is Ricci flat. On the other hand, using the Atiyah–Patodi–Singer index theorem for a manifold with boundaries it was concluded that the Taub–NUT metric makes no contribution to the axial anomaly [11–14].

In section 2 we verify explicitly that for generalized Taub–NUT metric the commutator (5) does not vanish and consequently there are gravitational anomalies.

In the next sections we compute the index of the Dirac operator for the generalized Taub–NUT metrics with the APS boundary condition and find these metrics do not contribute to the axial anomaly for not too large deformations of the standard Taub–NUT metric. This result stands in contrast with the gravitational anomalies for scalar fields discussed in section 2. The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this amounts to a continuous change in the metric and the boundary condition. However, for larger deformations of the metric there do appear discontinuities in the boundary condition and therefore the index presents jumps. Our formula for the index involves a computable number-theoretic quantity depending on the coefficients of the metric and on the radii of the domain.

In section 6 we propose some open problems in connection with unbounded domains. The last section contains concluding remarks.

2. Gravitational anomalies in generalized Taub–NUT spaces

The Euclidean Taub–NUT metric has lately attracted much attention in physics (see, e.g., [15, 2]). From the viewpoint of dynamical systems, the geodesic motion in Taub–NUT metric is known to admit a Kepler-type symmetry [16–19]. One can actually find the so-called Runge–Lenz vector as a conserved vector in addition to the angular momentum vector. As a consequence, all the bounded trajectories are closed and the Poisson brackets among the conserved vectors give rise to the same Lie algebra as the Kepler problem, depending on the energy.

Iwai and Katayama [3–6] generalized the Taub–NUT metric so that it still admits a hidden symmetry of Kepler type.

2.1. Generalized Taub–NUT spaces

The Euclidean Taub–NUT space is a special member of the family of four-dimensional manifolds equipped with the isometry group $G_{\rm iso} = SO(3) \otimes U(1)$. These geometries can be easily constructed defining the line elements in local charts with spherical coordinates $(r, \theta, \varphi, \chi)$; among them the first three are the usual spherical coordinates of the vector $\vec{x} = (x^1, x^2, x^3)$, with $|\vec{x}| = r$, while χ is the Kaluza–Klein extra-coordinate of this chart. The spherical coordinates can be associated with the Cartesian ones (x^1, x^2, x^3, x^4) where $x^4 = -\mu(\chi + \varphi)$ is defined using an arbitrary constant $\mu > 0$.

The group $SO(3) \subset G_{\rm iso}$ has three independent one-parameter subgroups, $SO_i(2), i = 1, 2, 3$, each one including rotations $\mathfrak{R}_i(\phi)$, of angles $\phi \in [0, 2\pi)$ around the axis i. With this notation any rotation $\mathfrak{R} \in SO(3)$ in the usual Euler parametrization reads $\mathfrak{R}(\alpha, \beta, \gamma) = \mathfrak{R}_3(\alpha)\mathfrak{R}_2(\beta)\mathfrak{R}_3(\gamma)$. Moreover, we can write $\vec{x} = \mathfrak{R}(\varphi, \theta, 0)\vec{x}_o$ where the vector $\vec{x}_o = (0, 0, r)$ is invariant under $SO_3(2)$ rotations which form its little group $(\mathfrak{R}_3\vec{x}_o = \vec{x}_o)$. The main point is to define the action of two arbitrary rotations, $\mathfrak{R} \in SO(3)$ and

$\mathfrak{R}_3 \in SO_3(2) \sim U(1)$, in the spherical charts, $(\mathfrak{R}, \mathfrak{R}_3) : (r, \theta, \varphi, \chi) \to (r, \theta', \varphi', \chi')$, such that

$$\mathfrak{R}(\varphi', \theta', \chi') = \mathfrak{R}\mathfrak{R}(\varphi, \theta, \chi)\mathfrak{R}_3^{-1}. \tag{6}$$

Hereby it results that the Cartesian coordinates transform under rotations $\mathfrak{R} \in SO(3)$ as

$$\vec{x} \to \vec{x}' = \mathfrak{R}\vec{x}, \qquad x^4 \to x'^4 = x^4 + h(\mathfrak{R}, \vec{x}),$$

where the function h is given in [20]. Thus, the vector $\vec{x}$ transforms according to an usual linear representation but the transformation of the fourth Cartesian coordinate is governed by a representation of $SO(3)$ induced by $SO_3(2)$ [20]. Furthermore, we observe that the 1-forms

$$\mathrm{d}\Omega(\varphi, \theta, \chi) = \mathfrak{R}(\varphi, \theta, \chi)^{-1}\, \mathrm{d}\mathfrak{R}(\varphi, \theta, \chi) \in so(2)$$

transform independently on $\mathfrak{R}$ as

$$(\mathfrak{R}, \mathfrak{R}_3) : \mathrm{d}\Omega(\varphi, \theta, \chi) \to \mathrm{d}\Omega(\varphi', \theta', \chi') = \mathfrak{R}_3\, \mathrm{d}\Omega(\varphi, \theta, \chi)\mathfrak{R}_3^{-1}$$

finding that, beside the trivial quantity ${\mathrm{d}s_1}^2 = \mathrm{d}r^2$, there are two types of line elements invariant under G_{iso},

$$\begin{aligned} {\mathrm{d}s_2}^2 &= -\langle \mathrm{d}\Omega(\varphi, \theta, \chi)^2\rangle_{33} = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2, \\ {\mathrm{d}s_3}^2 &= -\tfrac{1}{2}\, \mathrm{Tr}[\mathrm{d}\Omega(\varphi, \theta, \chi)^2] = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2 + (\mathrm{d}\chi + \cos\theta\, \mathrm{d}\varphi)^2. \end{aligned}$$

The conclusion is that the most general form of the line element invariant under G_{iso} is given by the linear combination $f_1(r)\, {\mathrm{d}s_1}^2 + f_2(r)\, {\mathrm{d}s_2}^2 + f_3(r)\, {\mathrm{d}s_3}^2$ involving three arbitrary functions of r, f_1, f_2 and f_3.

Here it is worth pointing out that the above metrics are related to the Berger family of metrics on 3-spheres [21]. These are introduced starting with the Hopf fibration $\pi_H : S^3 \to S^2$ that defines the vertical subbundle $V \subset TS^3$ and its orthogonal complement $H \subset TS^3$ with respect to the standard metric g_{S^3} on S^3. Denoting with g_H and g_V the restriction of g_{S^3} to the horizontal, respectively the vertical bundle, one finds that the corresponding line elements are ${\mathrm{d}s_H}^2 = \frac{1}{4}\, {\mathrm{d}s_2}^2$ and ${\mathrm{d}s_V}^2 = \frac{1}{4}\big({\mathrm{d}s_3}^2 - {\mathrm{d}s_2}^2\big)$. For each constant $\lambda > 0$ the Berger metric on S^3 is defined by the formula

$$g_\lambda = g_H + \lambda^2 g_V. \tag{7}$$

In what follows we restrict ourselves to the *generalized* Taub–NUT manifolds whose metrics are defined on $\mathbb{R}^4 - \{0\}$ by the line element

$$\begin{aligned} {\mathrm{d}s_K}^2 &= g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \\ &= f(r)(\mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\varphi^2) + g(r)(\mathrm{d}\chi + \cos\theta\, \mathrm{d}\varphi)^2 \end{aligned} \tag{8}$$

where the angle variables (θ, φ, χ) parametrize the sphere S^3 with $0 \leqslant \theta < \pi, 0 \leqslant \varphi < 2\pi, 0 \leqslant \chi < 4\pi$, while the functions

$$f(r) = \frac{a + br}{r}, \qquad g(r) = \frac{ar + br^2}{1 + cr + \mathrm{d}r^2} \tag{9}$$

depend on the arbitrary real constants a, b, c and d. This line element can be written in terms of the Berger metrics as

$${\mathrm{d}s_K}^2 = (ar + br^2)\left(\frac{\mathrm{d}r^2}{r^2} + 4\, {\mathrm{d}s_{\lambda(r)}}^2\right) \tag{10}$$

where ${\mathrm{d}s_{\lambda(r)}}^2 = (g_{\lambda(r)})_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$ and

$$\lambda(r) = \frac{1}{\sqrt{1 + cr + \mathrm{d}r^2}}. \tag{11}$$

If one takes the constants

$$c = \frac{2b}{a}, \qquad d = \frac{b^2}{a^2} \tag{12}$$

the generalized Taub–NUT metric becomes the original Euclidean Taub–NUT metric up to a constant factor.

By construction, the spaces with the metric (8) have the isometry group G_{iso} and, therefore, they must have four Killing vectors. The corresponding constants of motion in generalized Taub–NUT backgrounds consist of a conserved quantity for the cyclic variable χ

$$q = g(r)(\dot{\chi} + \cos\theta\dot{\varphi})$$

and the angular momentum vector

$$\vec{J} = \vec{x} \times \vec{p} + q\frac{\vec{x}}{r}, \qquad \vec{p} = f(r)\dot{\vec{x}}.$$

The remarkable result of Iwai and Katayama is that the generalized Taub–NUT space (8) admits a hidden symmetry represented by a conserved vector, quadratic in 4-velocities, analogous to the Runge–Lenz vector of the following form:

$$\vec{K} = \vec{p} \times \vec{J} + \kappa\frac{\vec{x}}{r}. \tag{13}$$

The constant κ involved in the Runge–Lenz vector (13) is $\kappa = -aE + \frac{1}{2}cq^2$ where the conserved energy E is

$$E = \frac{\vec{p}^2}{2f(r)} + \frac{q^2}{2g(r)}.$$

The components $K_i = k_i^{\mu\nu} p_\mu p_\nu$ of the vector $\vec{K}$ (13) involve three S–K tensors $k_i^{\mu\nu}$, $i = 1, 2, 3$ satisfying (1). Moreover, the components of the Runge–Lenz vector fill in the algebra of the angular momentum up to $o(4)$, $o(3, 1)$ or $e(3)$ algebras corresponding to different domains of the energy spectra, like in the case of the Kepler problem or the Euclidean Taub–NUT space [22, 23].

2.2. *The role of the K–Y tensors*

The gravitational quantum anomaly that does not exist in Ricci-flat manifolds can also be absent in manifolds which do not have this property if the S–K tensors have a special structure. We refer to the situation in which the S–K tensor $k_{\mu\nu}$ can be written as a product of K–Y tensors [8].

A K–Y tensor of valence 2 is an antisymmetric tensor $f_{\mu\nu}$ satisfying the Killing equation

$$f_{\mu(\nu;\lambda)} = 0. \tag{14}$$

Let us suppose that there exists a *square root* of the S–K tensor $k_{\mu\nu}$ of the form of a K–Y tensor $f_{\mu\nu}$ [8]:

$$k_{\mu\nu} = f_{\mu\rho} f_\nu{}^\rho. \tag{15}$$

In case this should happen, the S–K equation (1) is automatically satisfied and the integrability condition for any solution of (14) becomes

$$k^\rho_{[\mu} R_{\nu]\rho} = 0. \tag{16}$$

This relation implies the vanishing of the commutator (5) which means that the scalar quantum anomaly does not exist for the S–K tensors which admit a decomposition in terms of K–Y tensors.

In what follows we shall exemplify the role of the Killing–Yano tensors with regard to anomalies on the Euclidean Taub–NUT space and its generalizations. The (standard) Euclidean Taub–NUT space is a hyper-Kähler manifold possessing a triplet of covariantly constant K–Y tensors, $f^i, i = 1, 2, 3$. In addition, there exists a fourth K–Y tensor, f^Y, which is not covariantly constant. The presence of this last K–Y tensor is connected with the existence of the hidden symmetries of the Taub–NUT geometry which are encapsulated in three non-trivial S–K tensors and interpreted as the components of the so-called Runge–Lenz vector of geodesic motions in this space. All these S–K tensors are products of f^Y with f^i and, moreover, the manifold is Ricci flat since the metric tensor can also be expressed as a product of covariantly constant K–Y tensors through $f^{i\mu}{}_{(\alpha} f^j{}_{\beta)\mu} = -2\delta_{ij} g_{\alpha\beta}$ [22]. Obviously, for this metric there are no gravitational anomalies for scalar fields.

Concerning the generalized Taub–NUT metrics, as was done by Iwai and Katayama, it was proved that the extensions of the Taub–NUT metric do not admit K–Y tensors, even if they possess S–K tensors [24, 25]. The only exception is the original Taub–NUT metric which possesses four K–Y tensors of valence two.

Using the S–K tensor components of the Runge–Lenz vector (13) we can proceed to the evaluation of the quantum gravitational anomaly for the generalized Taub–NUT metric. A direct evaluation shows that the commutator (5) does not vanish.

To serve as a model for the evaluation of the commutator (5) involving the components of the S–K tensors corresponding to the Runge–Lenz vector (13), we limit ourselves to give only the components of the third S–K $k_3^{\mu\nu}$ tensor in spherical coordinates. Its non-vanishing components are

$$\begin{aligned}
k_3^{rr} &= -\frac{ar\cos\theta}{2(a+br)}\\
k_3^{r\theta} &= k_3^{\theta r} = \frac{\sin\theta}{2}\\
k_3^{\theta\theta} &= \frac{(a+2br)\cos\theta}{2r(a+br)}\\
k_3^{\varphi\varphi} &= \frac{(a+2br)\cot\theta\csc\theta}{2r(a+br)}\\
k_3^{\varphi\chi} &= k_3^{\chi\varphi} = -\frac{(2a+3br+br\cos(2\theta)\csc^2\theta}{4r(a+br)}\\
k_3^{\chi\chi} &= \frac{(a-a\,\mathrm{d}r^2+br(2+cr)+(a+2br))\cot^2\theta)\cos\theta}{2r(a+br)}.
\end{aligned}$$

Again, just to exemplify, we write down from the commutator (5) only the function which multiplies the covariant derivative D_r:

$$\begin{aligned}
&\frac{3r\cos\theta}{4(a+br)^3(1+cr+dr^2)^2}\{-2bd(2ad-bc)r^3+[3bd(2b-ac)-(ad+bc)(2ad-bc)]r^2\\
&\qquad+2(ad+bc)(2b-ac)r+a(2ad-bc)+(b+ac)(2b-ac)\}.
\end{aligned} \tag{17}$$

Recall that the commutator (5) vanishes for the standard Euclidean Taub–NUT metric. It is easy to see that the above expression (17) vanishes for all r if and only if the constants a, b, c, d are constrained by (12).

3. Index formulae on compact manifolds with boundary

Other quantum anomalies, namely axial ones, are connected with Dirac operators. For this reason the analysis of the Dirac operator for the standard Euclidean Taub–NUT space [22, 23] is extended here to the generalized Taub–NUT metrics. We interpret the axial anomaly as the index of the Dirac operator.

Atiyah, Patodi and Singer [26] discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. A serious drawback of this formula is the requirement that the metric and the operator be of 'product type' near the boundary. The generalized Taub–NUT metric is not of product type on any reasonable domain, so we need a different way of computing the index.

For Dirac operators on manifolds of the form $[l_1, l_2] \times M$, where M is closed, we give below another formula, in terms of the spectral flow of the family of Dirac operators over the slices $\{t\} \times M, l_1 \leqslant t \leqslant l_2$, without assuming that the metric is of product type near the boundary. For periodic families, this result appeared in [27]. The APS formula is used in the proof but one could probably give a direct proof.

3.1. The spectral flow

Let (M, g) be a closed Riemannian spin manifold of odd dimension with a fixed spin structure, Σ is the spinor bundle and D is the (self-adjoint) Dirac operator on M. Then D has discrete real spectrum accumulating towards $\pm\infty$. Moreover, the eta function

$$\eta(D, s) := \dim(\ker D) + \sum_{0 \neq \lambda \in \operatorname{Spec} D} |\lambda|^{-s} \operatorname{sign}(\lambda)$$

is holomorphic for $\Re(s) > \dim(M) - 1$ and extends meromorphically to $\mathbb{C}$. The point $s = 0$ is regular [27], and the value $\eta(D, 0)$ is by definition $\eta(D)$, the eta invariant of D. Let $g_t, l_1 \leqslant t \leqslant l_2$, be a smooth family of Riemannian metrics on M, and D_t the Dirac operator on M with respect to g_t and the fixed spin structure. Then

$$[l_1, l_2] \ni t \mapsto f(t) := \eta(D_t)/2 \in \mathbb{R}$$

is smooth modulo $\mathbb{Z}$, so $t \mapsto \exp(2\pi \mathrm{i} f(t)) \in S^1$ is smooth. By the homotopy lifting property, there exists a smooth lift $\tilde{f}$ of $\exp(2\pi \mathrm{i} f)$ to $\mathbb{R}$, the universal cover of S^1, uniquely determined by the condition $\tilde{f}(l_1) = f(l_1)$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp \circ 2\pi i"] \\ {[l_1, l_2]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "e^{2\pi i f}"'] & S^1 \end{tikzcd}$$

From the definition, it is evident that $\tilde{f}(t) - f(t) \in \mathbb{Z}$.

Definition 1. *The spectral flow of the family* $\{D_t\}_{l_1 \leqslant t \leqslant l_2}$ *is*

$$\operatorname{sf}(D_{l_1}, D_{l_2}) := f(l_2) - \tilde{f}(l_2).$$

This coincides with the original definition of the spectral flow for a path of self-adjoint Fredholm operators from [27, section 7], which heuristically counts the net number of eigenvalues crossing 0 in the positive direction. The spectral flow is clearly a path-homotopy invariant. Now the set of Riemannian metrics is convex inside the linear space of 2-tensors. Therefore the spectral flow of the pair $\left(D_{l_1}, D_{l_2}\right)$ is well defined using *any one*-parameter deformation of g_{l_1} into g_{l_2} and the associated path of Dirac operators.

3.2. A generalized APS index formula

Denote by $\mathcal{C}^\infty(M, \Sigma)$ the space of smooth spinor on M and let

$$\Pi^\pm : \mathcal{C}^\infty(M, \Sigma) \to \mathcal{C}^\infty(M, \Sigma)$$

be the spectral projections associated with D and the intervals $[0, \infty)$, respectively $(-\infty, 0]$. The spectrum of D is discrete. So in other words, if ϕ_T is an eigenspinor of D of eigenvalue T, then

$$\Pi^+(\phi_T) = \begin{cases} \phi_T & \text{if} \quad T \geqslant 0; \\ 0 & \text{otherwise;} \end{cases} \qquad \Pi^-(\phi_T) = \begin{cases} \phi_T & \text{if} \quad T \leqslant 0; \\ 0 & \text{otherwise.} \end{cases}$$

If X is a compact spin manifold-with-boundary of even dimension, let $\Sigma^\pm(X)$ denote the bundles of positive, respectively negative spinors. The spinor bundles $\Sigma(\partial X)$ and $\Sigma^\pm(X)_{|\partial X}$ over ∂X are canonically identified by the Clifford action of the unit normal vector field. We will need the following generalization of the Atiyah–Patodi–Singer index formula:

Theorem 2. *Let (X, g^X) be a compact spin Riemannian manifold with boundary, and*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^-) := \{\phi \in \mathcal{C}^\infty(X, \Sigma^+); \Pi^-(\phi|_{\partial X}) = 0\}.$$

Then the operator $D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^-) \to \mathcal{C}^\infty(X, \Sigma^-)$ is Fredholm, and

$$\operatorname{index}(D^+) = \int_X \hat{A}(g^X) + \int_{\partial X} T\hat{A} + \frac{1}{2}\eta(D_{\partial X})$$

where $T\hat{A}$, the transgression form of $\hat{A}$, depends on the 2-jets of g^X at ∂X.

Recall that the $\hat{A}$ form (whose cohomology class in top dimension on a closed $4k$-dimensional manifold gives the so-called $\hat{A}$ genus) is derived from the curvature tensor via the Chern–Weil construction:

$$\hat{A}(g) := \det\left(\frac{R/4\pi\mathrm{i}}{\sinh(R/4\pi\mathrm{i})}\right)^{1/2}.$$

Proof. The fact that D^+ is Fredholm is standard in the theory of elliptic boundary value problems, see, e.g., [28]. If the metric g^X was of product type near ∂X, then the Atiyah–Patodi–Singer formula [26] on X_t would read

$$\operatorname{index}(D^+) = \int_{X_t} \hat{A}(g^X) + \frac{1}{2}\eta(D_{\partial X}) \tag{18}$$

(we use the opposite orientation for ∂X as compared to [26]). In general we cannot expect such a product structure. In a collar neighbourhood defined by normal geodesic flow from ∂X, g^X takes the form

$$g^X = \mathrm{d}t^2 + g_t$$

for $0 \leqslant t < \epsilon$ (see [29]), where g_t is a smooth family of metrics on ∂X. So we first deform smoothly the metric g^X into a product metric near ∂X, keeping constant the metric at the boundary and outside the fixed collar neighbourhood, using a smooth function ψ:

$$h_s = \mathrm{d}t^2 + g_{\psi(s,t)}, \qquad \psi(s,t) = \begin{cases} t & \text{if} \quad s = 0 \quad \text{or} \quad t > \frac{3\epsilon}{4}; \\ 0 & \text{if} \quad t = 0; \\ 0 & \text{if} \quad s = 1 \quad \text{and} \quad t \leqslant \frac{\epsilon}{2}. \end{cases}$$

The index can be computed from the action of D^+ on Sobolev spaces:

$$D^+ : H^1(X, \Sigma^+, \Pi^-) \to L^2(X, \Sigma^-).$$

The spinor bundles for different metrics are canonically identified [29]. Since by construction the vector field $\partial/\partial t$ is normal to ∂X and of length 1 for all the metrics h_s, it follows that the projection Π^-, and hence also the space $H^1(X, \Sigma^+, \Pi^-)$, do not vary with s. Let D_s^+ be the Dirac operator corresponding to the metric h_s. Then the family of bounded operators

$$D_s^+ : H^1(X, \Sigma^+, \Pi^-) \to L^2(X, \Sigma^-)$$

is norm continuous, thus the index stays constant during the deformation. Therefore, we may compute $\operatorname{index}(D^+)$ using equation (18) for the metric h_1.

Next we relate the $\hat{A}$ forms using the transgression form. Consider the connection

$$\tilde{\nabla} := \mathrm{d}s \wedge \frac{\partial}{\partial s} + \nabla^s$$

on the bundle TX over $[0, 1] \times X$, where ∇^s is the Levi-Civita connection of the metric h_s. The curvature of $\tilde{\nabla}$ decomposes in

$$\tilde{R} = R^s + \mathrm{d}s \wedge \theta(s)$$

where $\theta(s)$ is defined by the above equality. Therefore

$$\hat{A}(\tilde{\nabla}) = \hat{A}(\nabla^s) + \mathrm{d}s \wedge \Theta(s)$$

and by inspection, $\Theta(s)$ depends on the 2-jets of the metric $g_{\psi(s,t)}$. Since $\hat{A}(\tilde{\nabla})$ is closed (like all characteristic forms), it follows that

$$\frac{\partial \hat{A}(\nabla^s)}{\partial s} = \mathrm{d}\theta(s). \tag{19}$$

Define

$$T\hat{A} := \int_0^1 \Theta(s)\,\mathrm{d}s.$$

By integrating (19) on $[0, 1]$, we get $\hat{A}(h_1) - \hat{A}(h_2) = \mathrm{d}T\hat{A}$. By Stokes's formula,

$$\int_X \hat{A}(h_1) - \int_X \hat{A}(g^X) = \int_{\partial X} T\hat{A}.$$

□

As defined, $T\hat{A}$ depends on the function ψ. For us the important conclusion is the next corollary.

Corollary 3. *Let $\{g_l^X\}_{l\in\mathbb{R}}$ be a smooth family of metrics on X, D_l^+ the associated family of Dirac operators on X and $D_{\partial X}^l$ the induced Dirac operator on ∂X. Then there exists a smooth function $u(l)$ such that*

$$\operatorname{index}\left(D_l^+\right) = u(l) + \tfrac{1}{2}\eta\left(D_{\partial X}^l\right).$$

Moreover, for $l_1 < l_2$,

$$\operatorname{index}\left(D_{l_2}^+\right) - \operatorname{index}\left(D_{l_1}^+\right) = \operatorname{sf}\left(D_{\partial X}^{l_1}, D_{\partial X}^{l_2}\right).$$

Proof. Clearly $\hat{A}\left(g_l^X\right)$ depends smoothly on l. From the construction, the transgression form is also clearly smooth in l once we fix the auxiliary function ψ. We define

$$u(l) := \int_X \hat{A}\left(g_l^X\right) + \int_{\partial X} T\hat{A}\left(g_l^X\right)$$

which by theorem 2 proves the first statement.

Using the notation from definition 1,

$$\begin{aligned}\operatorname{index}\big(D^+_{l_2}\big) - \operatorname{index}\big(D^+_{l_1}\big) &= u(l_2) - u(l_1) + f(l_2) - f(l_1)\\ &= u(l_2) - u(l_1) + \tilde f(l_2) - \tilde f(l_1)\\ &\quad + \operatorname{sf}\big(D^{l_1}_{\partial X}, D^{l_2}_{\partial X}\big).\end{aligned}$$

Thus the smooth function $u(l_2) - u(l_1) + \tilde f(l_2) - \tilde f(l_1)$ is integral valued, and so it vanishes identically since it does at $l = l_1$. The conclusion follows by setting $l = l_2$. □

3.3. Index theory on a cylinder

Let now g^X be a Riemannian metric on the cylinder $X := [l_1, l_2] \times M$. Endow X with the product orientation, so that $\{l_1\} \times M$ is positively oriented and $\{l_2\} \times M$ is negatively oriented inside X. Let D^+ be the chiral Dirac operator on X. For each $t \in [l_1, l_2]$ let g_t be the metric on M obtained by restricting g^X to $\{t\} \times M$. We denote by Σ_t the spinor bundle over (M, g_t) and by D_t, $\Pi^\pm_t$ the Dirac operator and the spectral projections with respect to the metric g_t.

As we mentioned above, there exist canonical identifications of the spinor bundle Σ_t with $\Sigma^\pm(X)_{|\{t\}\times M}$. Consequently it makes sense to denote by ϕ_t the restriction of a positive spinor from X to $\{t\} \times M$.

Theorem 4. *Let $X = [l_1, l_2] \times M$ be a product spin manifold with a smooth metric g^X as above. Set*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^-) := \big\{\phi \in \mathcal{C}^\infty(X, \Sigma^+);\ \Pi^+_{l_1}\phi_{l_1} = 0,\ \Pi^-_{l_2}\phi_{l_2} = 0\big\}.$$

Then

$$\operatorname{index}(D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^-) \to \mathcal{C}^\infty(X, \Sigma^-)) = \operatorname{sf}(D_{l_1}, D_{l_2}).$$

Note that the projection $\Pi^-_{l_2}$ equals $\Pi^+_{l_2}$ for the opposite orientation on $\{l_2\} \times M$, which is the one induced from X.

Proof. Deform the metric g^X in a neighbourhood of $\{l_1\} \times M$ to a product metric as in the proof of theorem 2. As explained there, this deformation does not change the index. The spectral flow is also unchanged (we noted that it depends only on the two metrics on the ends). For $l_1 < t \leqslant l_2$ let $X_t := [l_1, t] \times M \subset X$. Then corollary 3 gives

$$\begin{aligned}\operatorname{index}\big(D^+_t\big) &= u(t) + f(t) - f(l_1)\\ &= u(t) + \tilde f(t) - \tilde f(l_1) + \operatorname{sf}(l_1, t). \quad \text{by Def. 1}\end{aligned}$$

Note that both the $\hat A$ volume form and the transgression $T\hat A$, hence also $u(t)$, vanish for t near l_1 in the product region. Thus the smooth function $u(t) + \tilde f(t) - \tilde f(l_1)$ takes values in $\mathbb{Z}$, on the other hand both $u(t)$ and $\tilde f(t) - \tilde f(l_1)$ vanish at $t = l_1$, so $u(t) + \tilde f(t) - \tilde f(l_1)$ vanishes identically. The conclusion follows by setting $t = l_2$. □

Note that a similar statement concerning spectral boundary value problems appears in [30].

4. Harmonic spinors over Berger spheres

Since the cohomology groups of S^3 vanish in dimensions 1 and 2, there exists a unique spin structure on S^3. Let D_λ be the Dirac operator corresponding to the Berger metric g_λ defined in equation (7). Recall that D_λ is essentially self-adjoint (in L^2) with discrete spectrum.

Lemma 5. *For $\lambda < 2$, D_λ does not admit harmonic spinors.*

Proof. It is easy to compute the scalar curvature of g_λ. This is done for instance in [13]. Namely, $\kappa(g_\lambda)$ is constant on S^3, $\kappa(g_\lambda) = (4-\lambda^2)/12$. In particular $\kappa(g_\lambda)$ is positive for $\lambda < 2$. Lichnerowicz's formula proves then that $\ker D_\lambda = 0$. □

More generally, Hitchin [21] computed the eigenvalues of D_λ. In this paper we are only interested in eigenvalues close to 0. Let us recall Hitchin's result in this case.

Theorem 6 [21]. *Let*

$$\Lambda(\lambda) := \left\{(p,q) \in \mathbb{N}^{*2}; \lambda^2 = 2\sqrt{(p-q)^2 + 4\lambda^2 pq}\right\}.$$

Then

$$\dim\ker(D_\lambda) = N(\lambda) := \sum_{(p,q)\in\Lambda(\lambda)} p+q.$$

If $N(\lambda) > 0$ there exists $\epsilon > 0$ such that for $|t-\lambda| < \epsilon$, the 'small' eigenvalues of D_t are given by families

$$T(t,p,q) := \frac{t}{2} - \sqrt{\frac{(p-q)^2}{t^2} + 4pq}, \qquad (p,q) \in \Lambda(\lambda) \tag{20}$$

with multiplicity $p+q$.

In particular, harmonic spinors appear first for $\lambda = 4$ where the kernel of D_4 is two dimensional. Moreover, the set of those $\lambda \in (0,\infty)$ for which $N(\lambda) \neq 0$ is discrete. For $l > 0$ set

$$S(l) := \sum_{\lambda \leqslant l} N(\lambda). \tag{21}$$

Of course the sum is finite for finite l.

Corollary 7. *The spectral flow of the family $\{D_t\}_{t\in[l_1,l_2]}$ of Berger Dirac operators equals $S(l_2) - S(l_1)$.*

Proof. By differentiating equation (20) we see that the function $t \to T(t,p,q)$ is strictly increasing, so the spectral flow of the family $\{D_t\}$ across $t = \lambda$ is precisely $N(\lambda)$. □

5. The generalized Taub–NUT metric

Let us consider the generalized Taub–NUT metric $\mathrm{d}s_K^2$ on $\mathbb{R}^4\backslash\{0\} \simeq (0,\infty) \times S^3$ given by equation (10) in terms of the Berger metrics. We clearly need $a + br > 0$ for all $r > 0$ so we ask that $a > 0$, $b > 0$. Also $d > 0$ seems reasonable in order for the metric to be defined for large r, and even $c > -2\sqrt{d}$ so that $1 + cr + \mathrm{d}r^2 > 0$ for all $r > 0$. However, there seems to be no reason to ask $c \geqslant 0$, so $\lambda(r)$ may become large for certain values of r.

In mathematical terms, axial anomalies translate to Dirac operators with non-vanishing index. We are interested in the chiral Dirac operator on a annular piece of $\mathbb{R}^4\backslash\{0\}$. First set $X_{l_1,l_2} := [l_1, l_2] \times S^3 \subset \mathbb{R}^4\backslash\{0\}$ with the induced generalized Taub–NUT metric.

Theorem 8. *The index of D^+ over $\left(X_{l_1,l_2}, \mathrm{d}s_K^2\right)$ with the APS boundary condition is*

$$\operatorname{index}(D^+) = S(\lambda(l_2)) - S(\lambda(l_1))$$

where the function S is given by (21).

Proof. By theorem 4 the index is equal to the spectral flow of the pair of boundary Dirac operators. Now the metrics on the boundary spheres are constant multiples of the Berger metrics $g_{\lambda(l_1)}$, respectively $g_{\lambda(l_2)}$. The spectral flow of a path of conformal metrics (even with non-constant conformal factor) vanishes by the conformal invariance of the space of harmonic spinors [21]. Thus the spectral flow can be computed using the pair of metrics $g_{\lambda(l_1)}$ and $g_{\lambda(l_2)}$. The conclusion follows from corollary 7. □

It is a number-theoretic question to determine $S(\lambda)$ in general. We can give, however, some conditions which entail the vanishing of the index.

Corollary 9. *If $c > -\frac{\sqrt{15d}}{2}$ then the generalized Taub–NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof. The hypothesis implies that $\lambda(r) < 4$ for all $r > 0$. From the remark following theorem 6 we see that $S(\lambda(l_1)) = S(\lambda(l_2)) = 0$. □

We obtain as a particular case the vanishing of the index from [13]. Another case when the index vanishes is when l_1 and l_2 are either small or large enough so that both $\lambda(l_1)$ and $\lambda(l_2)$ are less than 4.

The singularity at the origin of the generalized Taub–NUT metric is removable, in the sense that there exists a smooth extension to $\mathbb{R}^4$.

Theorem 10. *For $l > 0$ let X_l be the ball $X_l := \{r \leqslant l\} \subset \mathbb{R}^4$, endowed with the generalized Taub–NUT metric $\mathrm{d}s_K^2$. Then*

$$\mathrm{index}(D^+) = S(\lambda(l)).$$

Proof. Deform the metric on X_l smoothly into the standard metric $\mathrm{d}s^2$ on the ball X_l. Now $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\sigma^2$ is a warped product near $r = l$, so we can further deform the warping factor to be constant near $r = l$. Let h_0 be the resulting metric and D_0^+ its Dirac operator. The restriction of h_0 to ∂X_l is a multiple of $g_1/4$, the standard metric on S^3. By corollaries 3 and 7,

$$\mathrm{index}(D^+) = \mathrm{index}\big(D_0^+\big) + S(\lambda(l))$$

since $S(1) = 0$. We use the APS index formula (18) to compute $\mathrm{index}\big(D_0^+\big) = 0$. Indeed, the eta invariant of the standard sphere vanishes since the spectrum is symmetric around 0, while the $\hat{A}$ volume form of a warped product metric vanishes by the conformal invariance of the Pontrjagin forms. □

6. Unbounded domains

There appear two other possibilities of constructing index problems for the metric $\mathrm{d}s_K^2$. First we have the mixed APS-L^2 boundary condition on $[l,\infty) \times S^3$; and secondly we have the L^2 index problem on $\mathbb{R}^4$.

The metric $\mathrm{d}s_K^2$ is of *fibred cusp* type at infinity in the sense of [31]. Indeed, with the change of variables $x = 1/r$ near $r = \infty$, we have

$$\mathrm{d}s_K^2 = (ax+b)\left(\frac{\mathrm{d}x^2}{x^4} + \frac{g_H}{x^2} + \frac{1}{d+cx+x^2}g_V\right). \tag{22}$$

It is impossible to present here Φ-operators and the associated Φ-calculus $\Psi_\Phi(\mathbb{R}^4)$; we refer the interested reader to [31–34]. The index of Dirac operators for *exact* Φ metrics was computed in [32] under a tameness assumption on the kernel of the family of vertical Dirac operators. Unfortunately, (22) is not exact in the sense of [32] because of the factor $d + cx + x^2$. A general but less precise index formula for fully elliptic Φ-operators was given in [33] and then improved in [34], where the case of a fibration over S^1 is studied in detail.

A priori it is not at all clear if D^+ is Fredholm in $L^2(\mathbb{R}^4)$, although from theorem 10, the limit as $l \to \infty$ of the index on X_l exists and equals 0. A general principle of Melrose's analysis of pseudodifferential algebras asserts that an operator in such an algebra is Fredholm on appropriate Sobolev spaces if and only if it is *fully elliptic*. Before explaining what this is, note that the corresponding statement for Φ-operators is proved in [31].

6.1. Fully elliptic Φ-operators

Let X denote the radial compactification of $\mathbb{R}^4$. There exists first a notion of principal symbol for Φ-operators, living on the Φ-cotangent bundle, a smooth extension of $T\mathbb{R}^4$ to X.

There exists additionally a 'boundary symbol' map called the normal operator, which is a star-morphism

$$\mathcal{N} : \Psi_\Phi(\mathbb{R}^4) \to \Psi_{\mathrm{sus}({}^\Phi T^* S^2)-\phi}(S^3)$$

with values in the *suspended algebra* [35], an algebra of parameter-dependent operators along the fibres of the Hopf fibration. A Φ-operator is called fully elliptic if both its principal symbol and its normal operator are invertible.

Theorem 11. *The Dirac operator on* $\left(\mathbb{R}^4, \mathrm{d}s_K^2\right)$ *is not fully elliptic.*

Proof. The principal symbol of D^2 is precisely the metric $\mathrm{d}s_K^2$, which extends to a Riemannian metric on ${}^\Phi T^*X$. This shows that D is elliptic.

Let $u : \mathbb{R}^4 \to (0, \infty)$ be a function which near $x = 0$ equals $ax + b$. Define a metric h on $\mathbb{R}^4$ conformal to $\mathrm{d}s_K^2$ by $h := \frac{\mathrm{d}s_K^2}{u}$. Then the Dirac operators of the metrics h and $\mathrm{d}s_K^2$ are related by [21, proposition 1.3]:

$$D_h = u^{5/4} D u^{-3/4}.$$

Now note that $u(x) = \sqrt{d} > 0$ for $x = 0$, and recall that the map $\mathcal{N}$ is multiplicative. Thus we see that the normal operators of D_h and D are simultaneously invertible. We focus in the rest of the proof on D_h.

We want to show that D_h is not fully elliptic, so we only look at the region $x \leqslant l$. Let $v(x) := \sqrt{d + cx + x^2}$, so that

$$h = \frac{\mathrm{d}x^2}{x^4} + \frac{g_H}{x^2} + \frac{1}{v^2(x)} g_V.$$

Let I, J, K be the three vector fields on S^3, viewed as the quaternion unit sphere, corresponding to the three distinguished complex structures. Let $V_j, 0 \leqslant j \leqslant 3$, be the following Φ-vector fields on X near the boundary:

$$V_0 := x^2 \partial_x, \qquad V_1 := v(x) I, \qquad V_2 := xJ, \qquad V_3 := xK.$$

These vector fields form an orthonormal frame p in ${}^\Phi TX$, parallel in the direction of V_0 (with respect to the Levi-Civita covariant derivative of the metric h). We use this frame (more precisely, one of its lifts $\tilde{p}$ to the spin bundle) to trivialize the spinor bundle. Denote by c^i the Clifford multiplication by V_i. After some computations, we get

$$\mathcal{N}(D_h)(0) = c^1(V_1 + \sqrt{\mathrm{d}} c^2 c^3)$$

in the above trivialization. Each integral curve $\mathcal{C}$ of V_1 has length $2\pi/\sqrt{d}$; let t be the arc-length parameter on $\mathcal{C}$. Let ψ be a spinor with

$$(V_1 + \sqrt{d} c^2 c^3)\psi = 0. \tag{23}$$

We can assume that ψ is a section of Σ^+, the other case is similar. The restriction of ψ to $\mathcal{C}$ is given by a curve

$$[0, 2\pi/\sqrt{d}) \ni t \mapsto \psi(t) \in \mathbb{C}^2,$$

where the two factors of $\mathbb{C}$ are the $\pm$i eigenspaces of c^2c^3. In other words, $\psi(t) = (\psi_+(t), \psi_-(t))$ with $c^2c^3\psi_\pm(t) = \pm \mathrm{i}\psi_\pm(t)$. Then equation (23) reduces to

$$\psi'(t) \pm \mathrm{i}\sqrt{d}\psi_\pm(t) = 0$$

and this equation does have solutions, namely $\psi_\pm(t) = \mathrm{e}^{\mp \mathrm{i}t\sqrt{d}}\psi(0)$. The point is that the solution is periodic of period equal to the length of $\mathcal{C}$. Equivalently, $\psi_\pm$ can be any smooth section in the complex line bundle over S^2 associated with the Hopf principal S^1-bundle $S^3 \to S^2$ and the ± 1 representations of S^1 on $\mathbb{C}$. □

Thus our Dirac operator is not Fredholm in $L^2(\mathbb{R}^4, \Sigma)$. However, it may still have a finite-dimensional kernel and cokernel. We leave open the question of determining the index in this case, but we conjecture it to be 0.

The same argument shows that the Dirac operator on $[l, \infty) \times S^3$ with mixed APS-L^2 boundary conditions is not Fredholm either. Again, we leave open the question of determining its index.

7. Concluding remarks

In spacetimes admitting Killing–Yano tensors there are additional supercharges in the dynamics of pseudo-classical spinning particles. On the other hand, there is a relationship between the absence of gravitational anomalies and the existence of K–Y tensors. For scalar fields, the decomposition (15) of S–K tensors in terms of K–Y tensors guarantees the absence of gravitational anomalies. Otherwise operators constructed from symmetric tensors are in general a source of anomalies proportional to the Ricci tensors.

However, for the axial anomaly the role of K–Y tensors is irrelevant. The axial anomaly vanishes for any generalized Taub–NUT metric where the constant c satisfies the inequality from corollary 9. For general constants c, the axial anomaly on annular domains or balls is given by a number-theoretic formula in terms of the radii (theorems 8, 10). As the radius of the ball increases to infinity, the axial anomaly on the ball becomes 0. Surprisingly, the corresponding Dirac operator on $\mathbb{R}^4$ is not Fredholm in L^2.

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